CVL746: Public Transportation Systems **Fall 2024**

Lecture 8: Random incidence and waiting time

Pramesh Kumar

8.1 Introduction

The interarrival time between two bus arrivals at a stop is called *headway*. If we record the bus arrival times at the stop as $T_0, T_1, \cdots T_k$, then headway can be calculated as:

$$\begin{aligned} H_1 =& T_1 - T_0 \\ H_2 =& T_2 - T_1 \\ &\vdots \\ H_k =& T_k - T_{k-1} \end{aligned}$$

This is an example of point process. We can characterize such a process using the family of joint pdf $f_{T_0,H_1,\cdots,H_k}(t_0, h_1, \cdots, h_k)$. For tractability, we consider a point process where the marginals pdfs follow the same distribution $f_H(h) = f_{H_1}(h_1) = \cdots = f_{H_k}(h_k)$. If $H_1, \cdots, H_k$ are independent as well then such process is called *renewal process*. A special case is *Poisson process* when $H_1, \cdots, H_k$ follow identical exponential distribution.

8.1.1 An example

Assume that the probability distribution of bus headways (interarrival times) is i.i.d. as below:

$$H = \begin{cases} 5, & \text{w.p. } 0.5 \\ 15, & \text{w.p. } 0.5 \end{cases}$$

Clearly, the average headway is $\mathbb{E}[H] = 5 \times 0.5 + 15 \times 0.5 = 10$ minutes. Should this be the average wait time? No. Consider the five bus headways as shown in Figure 8.1.1. Now you show up at the bus stop at a "random time".

Figure 8.1: Bus headways

$$\text{Prob (arriving during a 5 minute interval/headway)} = \frac{5}{5+15} = \frac{1}{4} \tag{8.1}$$

$$\text{Prob (arriving during a 15 minute interval/headway)} = \frac{15}{5+15} = \frac{3}{4} \tag{8.2}$$

as 15 minute interval occupies three times space as compared to 5 minute interval and the probability of arriving in any interval should depend on its length. Therefore, the expected length of the interarrival interval (or headway) during which you arrive is $\frac{1}{4} \times 5 + \frac{3}{4} \times 15 = 12.5$ minutes. This is true average wait time which is longer than $\mathbb{E}[H] = 10$ minutes. This is also called *random incidence paradox*. When you show up at the stop at random time, you are likely to fall on a larger headway due to its length. This is the problem of *random incidence* since an individual is incidence to the process at a random time (Larson and Odoni [1981]).

8.2 Average wait time

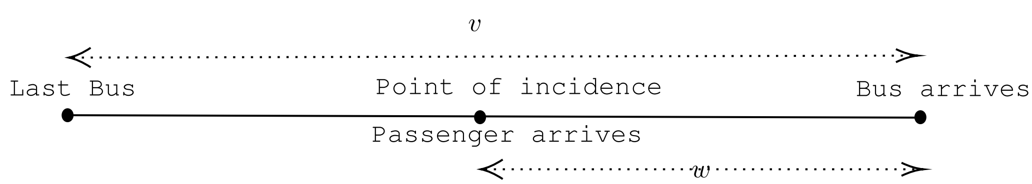

Figure 8.2: Random incidence process

The gap (headway) in which a potential bus passenger arrives has length equal to the sum of two time intervals.

1. Time between arrival of the most recent bus and arrival of passenger
2. Time between passenger's arrival and the arrival of next bus W (this is the time the passenger has to wait)

Let's call that the headway in which the passenger arrives as v (length of the interarrival gap entered by the passenger). Taking the inspiration from the example in Section 8.1.1, the probability of the gap entered by the incidence process will be directly proportional to the length of the gap (headway) v and the probability of occurrence of such gaps (of this particular length v) $f_H(v)$.

$$f_V(v) \propto v f_H(v) \implies f_V(v) = C v f_H(v) \tag{8.3}$$

where, C is the proportionality constant. We know that $\int_0^\infty f_V(v)dv = 1$ since $f_V(v)$ is a pdf.

$$\int_0^\infty f_V(v)dv = C \int_0^\infty v f_H(v) dv \tag{8.4}$$

$$\implies 1 = C\mathbb{E}[H] \tag{8.5}$$

$$\implies C = \frac{1}{\mathbb{E}[H]} \tag{8.6}$$

Therefore, the pdf of V is given by

$$f_V(v) = \frac{v f_H(v)}{\mathbb{E}[H]} \tag{8.7}$$

Now given the gap of length v, the passenger is equally likely to be anywhere in the gap (because of the random arrival). Given v, the time until the completion of the gap (when bus arrives), also called the wait time follows the uniform distribution whose pdf is given by:

$$f_{W|V}(w|v) = \frac{1}{v}, 0 \leq w \leq v \tag{8.8}$$

Using (8.7) and (8.8), the joint pdf of W, V is given by

$$\begin{aligned} f_{W,V}(w,v) =& f_{W|V}(w|v) f_V(v) && (8.9) \\ =& \frac{1}{v} \times \frac{v f_H(v)}{\mathbb{E}[H]}, 0 \leq w \leq v < \infty && (8.10) \end{aligned}$$

The marginal of W can be obtained by simply integrating out V,

$$f_W(w) = \int_w^\infty \frac{f_H(v)}{\mathbb{E}[H]} dv \tag{8.11}$$

which can be expressed in terms of cdf of H, $F_H(h)$,

$$f_W(w) = \frac{1 - F_H(w)}{\mathbb{E}[H]}, w \geq 0 \tag{8.12}$$

We can compute $\mathbb{E}[W]$ as below:

$$\mathbb{E}[W] = \int_0^\infty \mathbb{E}[W|v] f_V(v) dv$$

We know that given $V = v$, W is uniformly distributed between 0 and v, therefore $\mathbb{E}[W|v] = \frac{v}{2}$. Using (8.7), we have

$$\mathbb{E}[W] = \int_0^\infty \frac{v}{2} \times \frac{v f_H(v)}{\mathbb{E}[H]} dv$$

$$\boxed{\mathbb{E}[W] = \frac{\mathbb{E}[H^2]}{2\mathbb{E}[H]} = \frac{\sigma_H^2 + (\mathbb{E}[H])^2}{2\mathbb{E}[H]}}$$

For perfectly scheduled systems, $\sigma_H^2 = 0$, therefore $\mathbb{E}[W] = \frac{\mathbb{E}[H]}{2}$. More is the variability in headway, more is the wait time.

8.3 Cases

8.3.1 Case 1

Buses maintain the perfect headway, i.e., they are always $\tilde{H}$ minutes apart. Then,

$$F_H(w) = \begin{cases} 0, & \text{if } w < \tilde{H} \\ 1, & \text{if } w \geq \tilde{H} \end{cases}$$

Computing $f_W(w)$ using (8.12),

$$f_W(w) = \begin{cases} \frac{1}{\tilde{H}}, & \text{if } 0 \leq w \leq \tilde{H} \\ 0, & \text{otherwise} \end{cases}$$

which is uniformly distributed. In this case $\mathbb{E}[W] = \frac{\tilde{H}}{2}$ as we might expect.

8.3.2 Case 2

Suppose bus arrival is Poisson process with mean λ, so that the headway follows exponential distribution with rate $\frac{1}{\lambda}$. Hence, $F_H(w) = 1 - e^{-\lambda w}, w \geq 0$ and

$$f_W(w) = \frac{1 - (1 - e^{-\lambda w})}{\frac{1}{\lambda}} = \lambda e^{-\lambda w}, w \geq 0$$

The wait time follows exponential distribution with rate λ.

8.4 Common bus lines problem

In a city with overlapping transit routes, a passenger is faced with *common bus lines problem.* Chriqui and Robillard [1975] argue that the passenger will select a subset of common lines (also called *attractive set*) so as to minimize her total expected travel time, which is comprised of expected waiting time and expected travel time. Once this subset is selected, the passenger will take the first arriving vehicle among the selected bus routes.

Assume that the headway of individual bus routes $H_1, \cdots, H_k$ in the attractive set are independent and follows exponential distribution with rate $f_1, \cdots, f_k$. Then, wait time of individual routes $W_1, \cdots, W_k$ also follow exponential distribution with rate $f_1, \cdots, f_k$ (see Section **??**).

Prob(boarding route i) = Prob(waiting for i is less than other routes) = Prob($W_i < W_j, \forall j \neq i$)
$= \int_0^\infty \text{Prob}(W_j > W_i | W_i, \forall j \neq i) f_{W_i}(w_i) dw_i = \int_0^\infty \Pi_{j \neq i} \text{ Prob}(W_j > W_i | W_i) \ f_{W_i}(w_i) dw_i$
$= \int_0^\infty \Pi_{j \neq i} \left(e^{-f_j w_i}\right) f_i e^{-f_i w_i} dw_i = \int_0^\infty f_i e^{-(\sum_p f_p) w_i} dw_i = \frac{f_i}{\sum_p f_p}$.

Prob(wait time at the bus stop conditional on boarding route i) $f_i e^{-(\sum_p f_p) w}, w \geq 0$

Distribution of wait time at the stop is given by $f_W(w) = \sum_i \text{Prob } (w | \text{boarding } i) = \sum_i f_i e^{-(\sum_p f_p) w}, w \geq 0$.

Expected wait time at the stop considering the attractive set of lines
$= \mathbb{E}[W] = \int_0^\infty w (\sum_i f_i e^{-(\sum_p f_p) w}) dw = \frac{1}{\sum_p f_p}$.

Acknowledgment

The scriber would like to thank Dr. Niao He for the LaTeXtemplate.

References

Claude Chriqui and Pierre Robillard. Common bus lines. *Transportation science*, 9(2):115–121, 1975.

Richard C Larson and Amedeo R Odoni. *Urban operations research*. Number Monograph. Dynamic Ideas, 2nd editio edition, 1981.